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Linear Approximating Processes with Limited Oscillation

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1. INTRODUCTION

Korovkin's well-known theorem [10, pp. 14–17] asserts that the convergence of a sequence of positive linear operators on $C[a, b]$ to itself, to the identity, depends only on its convergence on three test functions. Several authors [2, 3, 8] gave the rate of convergence in terms of the moduli of continuity of the function approximated and the rate of convergence on three test functions. It is well known that for positive linear operators the order of magnitude of the rate of convergence improves with the smoothness of the functions up to $C^2[a, b]$, whereas it does not improve with smoother classes of functions. To generalize the above and discuss rates of convergence that depend on higher degrees of smoothness, one has to consider nonpositive operators and more test functions.

A linear functional on $C[a, b]$, $L_n(f, x)$ can be written as

$$L_n(f, x) = \int_a^b f(t)\, d\alpha_{n,x}(t) \qquad \text{where} \quad \alpha_{n,x}(t) \in B.V.[a, b]. \tag{1.1}$$

The oscillation bound, $\tau_n(x)$ or $\tau_n(S)$, is given by:

$$\tau_n(x) = \inf\{\tau \mid \alpha_{n,x}(t) \text{ is monotonic in } [a, b]\backslash[x - \tau, x + \tau]\}; \tag{1.2}$$

$$\tau_n(S) = \sup\{\tau_n(x) \mid x \in S\} \qquad S \subset [a, b]. \tag{1.3}$$

To illustrate our result we present the following special but quite applicable case:

THEOREM 1.1. *Suppose*: (1) $u_i(x)$ $i = 0,..., 2m$ *is an extended Tchebicheff system* (E.T.S.) *of order* $2m + 1$ [9, p. 6];

(2) *$L_n(f, x)$ are linear bounded operators on $C[a, b]$ satisfying*

$$\| L_n(u_i, \cdot) - u_i(\cdot)\| \leqslant \sigma_n^{2m} = o(1) \qquad n \to \infty;$$

(3) *$\tau_n([a, b])$ given by* (1.3) *satisfies $\tau_n([a, b]) \leqslant \sigma_n$; then*

$$\| L_n(f, \cdot) - f(\cdot)\| \leqslant K_1\sigma_n^{2m} + K_2\omega_{2m}(f, \sigma_n) \tag{1.4}$$

where $\omega_{2m}(f, h)$ is the 2mth *modulus of smoothness.*

One should note that for periodic convolution operators the same crucial quantity of $\tau_n(x)$ was achieved by Butzer, Nessel, and Sherer [1] while getting lower estimates, i.e., in a direction opposite to that of the present paper.

2. The Main Result

In this section we state our main result and derive from it the applicable Theorem 1.1. We recall that $u_0(x),..., u_{2m}(x)$ is an E.T.S. of order $2m + 1$ (see Ref. 9, p. 6) if $u_i(x) \in C^{2m}[a, b]$, $x_0 \leqslant \cdots \leqslant x_j < x_{j+1} \leqslant \cdots \leqslant x_{2m}$ and $\det \| w_{ij} \| > 0$ where

$$w_{ij} = \begin{cases} u_i(x_j) & \text{if } \; x_{j-1} \quad x_j \\ u_i^{(k)}(x_j) & \text{if } \; x_j = x_{j-k} \quad x_j \neq x_{j-k-1} . \end{cases}$$

THEOREM 2.1. *Suppose*: (1) *$u_i(x)$ $0 \leqslant i \leqslant 2m$ is an* E.T.S. *of order* $2m + 1$ *on* $[a, b]$;

(2) *$L_n(f, x)$ is a linear functional on $C[a, b]$ and $\tau_n(x) \equiv \tau_n$ given by* (1.2);

(3) $| L_n(u_i, x) - u_i(x)| \leqslant \psi_n$ $0 \leqslant i \leqslant 2m$; *then for every $f \in C[a, b]$ and* $0 < h < \min(b - x, x - a)$ *we have*

$$\begin{aligned} | L_n(f, x) - f(x)| \leqslant C_1\{(1 + \| L_n(x)\| + h^{-2m}\psi_n \\ + (h^{-1}\tau_n(x))^{2m} \| L_n(x)\|) w_{2m}(f, h) \\ + C_2(\psi_n + \tau_n^{2m} \| L_n(x)\|) \|f\| \end{aligned} \tag{2.1}$$

where $\| L_n(x)\|$ is the norm of (1, 1) *as a functional.*

Theorem 1.1 follows Theorem 2.1 if we recall that $\| L_n(x)\| \leqslant \| L_n \|$ and substitute $\psi_n = \sigma_n^{2m}$, $\tau_n \leqslant A\sigma_n$ (both for all $x \in [a, b]$) as well as $h = \sigma_n$. One can note also that Theorem 1.1 implies the results for positive linear operators discovered earlier [2, 3, 8].

Remark 2.1. It is obvious from the above that if for $S \subset [a, b]$ $\tau_n(S) \leqslant A\sigma_n$ and $L_n(f, x)$ are bounded linear transformations from $C[a, b]$ to $C(S)$, then (1.4) is true with $C(S)$ norm instead of $C[a, b]$ norm.

3. Some Lemmas

We shall need the following lemmas in the proof of Theorem 2.1.

Lemma 3.1. *The function*

$$V(x;t)=\begin{vmatrix} u_0(x) & u_1(x) & \cdots & u_{2m}(x) \\ \vdots & \vdots & & \vdots \\ u_0^{(2m-1)}(x) & u_1^{(2m-1)}(x) & \cdots & u_{2m}^{(2m-1)}(x) \\ u_0(t) & u_1(t) & \cdots & u_{2m}(t) \end{vmatrix} \tag{3.1}$$

satisfies the inequalities

$$A_1(x-t)^{2m} \leqslant V(x,t) \leqslant A_2(x-t)^{2m}. \tag{3.2}$$

Proof. Since $(\partial^k V/\partial t^k)|_{t=x}=0$ for $k=0,\dots,2m-1$ and $(\partial^{2m}V/\partial t^{2m})|_{t=x}>0$, (3.2) is valid for $|t-x|<\delta$ for some $\delta>0$. Since u_i is an E.T.S., $V(x,t)>0$ for $x\neq t$, and therefore (3.2) is valid with some constants for every such t. Compactness of $[a,b]\backslash(x-\delta,x+\delta)$ and continuity of $V(x,t)$ limit the above constants, which completes the proof.

Lemma 3.2. *For every $\phi\in C^{(2m)}[a,b]$ we have*

$$\|\phi^{(k)}\| \leqslant C(k)(\|\phi\|+\|\phi^{(2m)}\|). \tag{3.3}$$

Proof. The norm here is in the sense of $C[a,b]$ and therefore L_∞. This result is well known and extensively generalized (see, for example, Ref. 5, p. 5).

Define for $\phi\in C^{2m}[a,b]$ the "Taylor u polynomial"

$$U(\phi;x,t)=\sum_{k=0}^{2m} A_k(\phi;x)\,u_k(t) \tag{3.4}$$

satisfying

$$\left.\frac{\partial^j U}{\partial t^j}\right|_{t=x}=\phi^{(j)}(x) \qquad j=0,\dots,2m. \tag{3.5}$$

We may so define the above for u_i that are an E.T.S. of order $2m+1$.

Lemma 3.3. *For $U(\phi,x,t)$ defined by* (3.4) *and* (3.5) *we have*

$$|A_k(\phi,x)| \leqslant \gamma(k)(\|\phi\|+\|\phi^{(2m)}\|) \qquad k=0,\dots,2m \tag{3.6}$$

and

$$|\phi(t)-U(\phi;x,t)| \leqslant \gamma(\|\phi\|+\|\phi^{(2m)}\|)(x-t)^{2m}. \tag{3.7}$$

Proof. The determinant of the linear system given by (3.4) and (3.5) is $(\partial^{2m} V/\partial t^{2m})\,|_{t=x}$ which is positive, and therefore $|A_k(\phi, x)| \leqslant M(k) \sum_{j=1}^{2m} \|\phi^j\|$, which by Lemma 3.2 implies (3.6). The inequality (3.7) follows from (3.5) and (3.6).

LEMMA 3.4. *Under the conditions of Theorem* 2.1 *we have*

$$\int_a^b (x-t)^{2m}\,|d\alpha_{n,x}(t)| \leqslant M_1\psi_n + M_2\,\|L_n(x)\|\,(\tau_n(x))^{2m}. \tag{3.8}$$

Proof. Clearly (where for $t < a$ $\alpha_{nx}(t) = \alpha_{nx}(a)$ and for $t > b$ $\alpha_{nx}(t) = \alpha_{nx}(b)$)

$$\int_{x-\tau_n}^{x+\tau_n} (x-t)^{2m}\,|d\alpha_{nx}(t)| \leqslant \tau_n^{2m} \int_a^b |d\alpha_{nx}(t)| \leqslant \tau_n^{2m}\,\|L_n(x)\|. \tag{3.9}$$

On $J_n \equiv [a, b]\backslash[x-\tau_n, x+\tau_n]$ $\alpha_{n,x}(t)$ is monotonic, and therefore using Lemma 3.1 we have

$$\begin{aligned}\int_{J_n} (x-t)^{2m}\,|d\alpha_{nx}(t)| &= \left|\int_{J_n} (x-t)^{2m}\,d\alpha_{nx}(t)\right| \leqslant A_1^{-1}\left|\int_{J_n} V(x,t)\,d\alpha_{nx}(t)\right| \\ &\leqslant A_1^{-1}\left(\left|\int_a^b V(x,t)\,d\alpha_{n,x}(t)\right| + \int_{x-\tau_n}^{x+\tau_n} V(x,t)\,|d\alpha_{nx}(t)|\right) \\ &\leqslant A_1^{-1}\,|L_n(V(x,\cdot), x)| \\ &\quad + A_1^{-1}A_2 \int_{x-\tau_n}^{x+\tau_n} (x-t)^{2m}\,|d\alpha_{nx}(t)|.\end{aligned}$$

Since $V(x, x) = 0$ we have, using (3.1) and Condition (3)

$$|L_n(V(x,\cdot); x) - V(x,x)| \leqslant \text{const.}\ \psi_n\,.$$

Also

$$\int_{x-\tau_n}^{x+\tau_n} (x-t)^{2m}\,|d\alpha_{nx}(t)| \leqslant \tau_n^{2m} \int_a^b |d\alpha_{nx}(t)| = \tau_n^{2m}\|L_n(x)\|,$$

which completes the proof of the lemma.

4. PROOF OF THE MAIN RESULT

We first prove Theorem 2.1 for functions in $C^{2m}[a, b]$.

LEMMA 4.1. *Under the assumptions of Theorem* 2.1 *we have for* $\phi \in C^{2m}[a, b]$

$$|L_n(\phi, x) - \phi(x)| \leqslant B(\psi_n + \|L_n(x)\|\,\tau_n^{2m})(\|\phi\| + \|\phi^{(2m)}\|). \tag{4.1}$$

Proof. Recalling the "Taylor u polynomial" (3.4) we write

$$\begin{aligned} &| L_n(\phi, x) - \phi(x)| \\ &\quad \leqslant | L_n(\phi; x) - L_n(U(\phi; x, \cdot), x)| + | L_n(U(\phi; x, \cdot)x) - U(\phi; x, x)| \\ &\qquad + | U(\phi; x, x) - \phi(x)| \equiv J_1 + J_2 + J_3 . \end{aligned}$$

Using (3.7) and (3.8) consecutively, we have

$$\begin{aligned} J_1 &\leqslant \gamma(\| \phi \| + \| \phi^{(2m)} \|) \int_a^b | t - x |^{2m} | d\alpha_{nx}(t)| \\ &\leqslant \gamma(\| \phi \| + \| \phi^{(2m)} \|)(M_1\psi_n + M_2 \| L_n(x)\| | \tau_n(x)|^{2m}. \end{aligned}$$

Using (3.4), (3.6) and Assumption (3) of Theorem 2.1, we have $J_2 \leqslant (\| \phi \| + \| \phi^{(2m)} \|)(\sum_{k=0}^{2m} \gamma(k)) \psi_n$. Clearly from (3.4) and (3.5) $U(\phi, x, x) = \phi(x)$ or $J_3 = 0$. We complete the proof combining estimates for J_1, J_2, and J_3. To derive Theorem 2.1 from Lemma 4.1, we shall need the following general lemma of G. Freud and V. Popov [4].

LEMMA 4.2. *For an arbitrary $f \in C[a, b]$ and $0 < h < 1$ there exists $\phi_h \in C^{2m}[a, b]$ for which we have*

$$\|f - \phi_h \| \leqslant L_1\omega_{2m}(f, h) \tag{4.2}$$

and

$$\| \phi_h^{(2m)} \| \leqslant L_2h^{-2m}\omega_{2m}(f, h). \tag{4.3}$$

Proof of Theorem 2.1. We approximate f by ϕ_h as follows:

$$\begin{aligned} &| L_n(f, x) - f(x)| \\ &\quad \leqslant | L_n(f, x) - L_n(\phi_h , x)| + | L_n(\phi_h , x) - \phi_h(x)| + | \phi_h(x) - f(x)| \\ &\quad \equiv I_1 + I_2 + I_3 . \end{aligned}$$

We estimate using (4.2) $I_3 \leqslant L_1\omega_{2m}(f, h)$ and $I_1 \leqslant \| L_n(x)\| L_1\omega_{2m}(f, h)$. Using (4.3) and Lemma 4.1, we have

$$\begin{aligned} I_2 &\leqslant B(\psi_n + \| L_n(x)\| \tau_n(x)^{2m})(\| \phi_h \| + \| \phi_k^{2m} \|) \\ &\leqslant B(\psi_n + \| L_n(x)\| \tau_n(x)^{2m})(\|f\| + L_1\omega_{2m}(f, h) + L_2h^{-2m}\omega_{2m}(f, h)). \end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, we complete the proof of Theorem 2.1.

5. Applications to Convolution Type Operators

A. *The first application is actually the one that motivated this research.*

Define the operator $K_n(f, x)$ by

$$K_{n,r}(f, x) = \int_{-\infty}^{\infty} f(x + rt)\, d\mu_n(t), \qquad \int_{-\infty}^{\infty} d\mu_n(t) = 1, \tag{5.1}$$

$$K_n(f, x) = \binom{2m}{m}^{-1} \sum_{0<|r|\leqslant m} (-1)^{r+1} \binom{2m}{r+m} K_{n,r}(f; x). \tag{5.2}$$

We observe that $K_n(f, x) = \int_{-\infty}^{\infty} f(x + t)\, d\nu_n(t)$ *where*

$$\nu_n(t) = \binom{2m}{m}^{-1} \sum_{0<|r|<m} (-1)^{r+1} \binom{2m}{r+m} \mu_n\left(\frac{t}{r}\right). \tag{5.3}$$

We can also see that for $u_j(x) = x^j$

$$u_j(x) - K_n(u_j, x) = \begin{cases} 0 & j = 0,\ldots, 2m-1 \\ (2m)! \binom{2m}{m}^{-1} \int_{-\infty}^{\infty} t^{2m}\, d\nu_n(t) \equiv \sigma_n^{2m} & j = 2m. \end{cases} \tag{5.4}$$

As an application of Theorem 1.1 and 2.1 the following theorem is valid for f with compact support. However, the restriction of having compact support can be dropped (using the convolution structure) following the proof of 2.1.

THEOREM 5.1. *Suppose*: (a) $K_n(f, x)$, $\nu_n(t)$ *and* σ_n *are given by* (5.1), (5.2), (5.3) *and* (5.4); (b) $\int |d\nu_n(t)| \leqslant M < \infty$, *and* (c) $\nu_n(t)$ *is monotonic in* $R\backslash[-A\sigma_n, A\sigma_n]$; *then*

$$| K_n(f, x) - f(x)| \leqslant M[\omega_{2m}(f, \sigma_n) + \sigma_n^{2m}]. \tag{5.5}$$

B. *Application with a trigonometric test system.*

The De La Vallée Poussin operator is defined (see, e.g., Natanson [7, p. 20]) by:

$$V_n(f; x) = \alpha_n \int_{-\pi}^{\pi} f(t) \cos^{2n} \frac{t-x}{2}\, dt \qquad \alpha_n = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)}. \tag{5.6}$$

Define

$$V_n^{(2)}(f, x) = 2V_{2n-1}(f; x) - V_{n-1}(f; x). \tag{5.7}$$

THEOREM 5.2. *For every* 2π *periodic continuous* $f(x)$

$$\| V_n^{(2)}(f, x) - f(x)\| \leqslant C(n^{-2} + \omega_4(f, n^{-1/2})). \tag{5.8}$$

Proof. Let our Tchebicheff system be 1, $\cos t$, $\sin t$, $\cos 2t$ and $\sin 2t$. Elementary calculations yield:

$$V_n^{(2)}(u_i, x) = u_i(x) \text{ for } i = 0, 1, 2 \text{ where } u_0(t) = 1,\ u_1(t) = \cos t, \text{ and } u_2(t) = \sin t; \quad (5.9)$$

and

$$V_n^{(2)}(u_i, x) = u_i(x) + O(1/n^2) \text{ for } i = 3, 4 \text{ where } u_3(t) = \cos 2t \text{ and } u_4(t) = \sin 2t. \quad (5.10)$$

Following (5.7), we have $\| V_n^{(2)}(f, x)\| \leqslant 3 \|f\|$. The kernel of $V_n^{(2)}$ is $v_n^{(2)}(t) = 2\alpha_{2n-1} \cos^{4n-2}(t/2) - \alpha_{n-1} \cos^{2n-2}(t/2)$, and since $1 \leqslant 2\alpha_{2n-1}/\alpha_n \leqslant 4$ combined with the estimate $\cos^{2n} t < \cos^{2n}(K/\sqrt{n}) < [1 - (K^2/4n)]^{2n} < 2e^{-K^2/2}$ for $|t| > (K/\sqrt{n})$ implies that $v_n^{(2)}(t)$ is negative for $|t| > (K/\sqrt{n})$ if K is large enough, our theorem follows Theorem 1.1.

Remark. For $V_n^{(3)}(f, x)$ given by

$$V_n^{(3)}(f, x) = (8/3)\, V_{4n-1}(f, x) - 2V_{2n-1}(f, x) + (1/3)\, V_{n-1}(f, t),$$

$$\| V_n^{(3)}(f, \cdot) - f(\cdot)\| \leqslant K\left(n^{-3} + \omega_6\left(f, \frac{1}{\sqrt{n}}\right)\right). \quad (5.11)$$

6. Modified Bernstein Polynomials

The well known Bernstein polynomials are given by

$$B_n(f, x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k}{n}\right) \quad \text{where} \quad p_{nk}(x) = \binom{n}{k} x^k(1 - x)^{n-k}. \quad (6.1)$$

We shall define the modified Bernstein polynomial

$$B_n^{(2)}(f, x) = 2B_{2n}(f, x) - B_n(f, x) \quad (6.2)$$

and it will serve as the following nonconvolution application to our theorem.

THEOREM 6.1. *Let $\delta > 0$ be a fixed number and $f \in C[0, 1]$; then for* $S = [\delta, 1 - \delta]$,

$$\| B_n^{(2)}(f, \cdot) - f(\cdot)\|_{C(S)} \leqslant K(n^{-2} + \omega_4(f, n^{-1/2})). \quad (6.3)$$

Proof. One can easily verify

$$\| x^i - B_n^{(2)}(t^i; x)\|_{C[0,1]} = O(n^{-2}) \qquad i = 0, 1, 2, 3, 4. \quad (6.4)$$

Since we cannot apply Theorem 2.1 directly to $B_n^{(2)}(f, x)$ we will apply it to $L_n(f, x)$ given by

$$L_n(f, x) = 2B_{2n}(f, x) - B_n{}^*(f, x) \tag{6.5}$$

where

$$B_n{}^*(f, x) = \sum_{k=0}^{n} \frac{1}{3}\left(f\left(\frac{k-1/2}{n}\right) + f\left(\frac{k}{n}\right) + f\left(\frac{k+1/2}{n}\right)\right) p_{nk}(x) \tag{6.6}$$

where $f[-(1/2n)] = f(0)$ and $f[1 + (1/2n)] = f(1)$. From the above definition

$$| B_n{}^*(f; x) - B_n(f, x)| \leqslant \frac{1}{3}\,\omega_2\left(f; \frac{1}{2n}\right) + \frac{1}{3}\left|f\left(\frac{1}{2n}\right) - f(0)\right|(1-x)^n$$
$$+ \frac{1}{3}\left|f(1) - f\left(1 - \frac{1}{2n}\right)\right| x^n.$$

For a fixed δ, $\delta \leqslant x \leqslant 1 - \delta$, this implies

$$| B_n{}^*(f, x) - B_n(f, x)| \leqslant \frac{1}{3}\,\omega_2\left(f, \frac{1}{2n}\right) + Cn^{-2}\,\|f\|. \tag{6.7}$$

Equation (6.7) implies

$$\| L_n(t^i, x) - x^i \|_{C(S)} \leqslant Mn^{-2}, \qquad i = 0, 1, 2, 3, 4. \tag{6.8}$$

We shall need the following lemma:

LEMMA 6.2. *There exists A large enough, such that for $| x - (k/n)| \geqslant An^{-1/2}$ and $\delta < x < 1 - \delta$*

$$p_{n,k}(x) \geqslant 6 \max\{ p_{2n,2k+i}(x) \mid i = -1, 0, 1\}(p_{2n,2n+1}(x) = p_{2n,-1}(x) = 0). \tag{6.9}$$

This lemma will imply the monotonicity of $\alpha_{n,x}(t)$ for $| t - x | \geqslant A(1/\sqrt{n})$, and since $\psi_n \sim n^{-2}$, we will have

$$\| L_n(f, x) - f(x)\|_{C(S)} \leqslant A\left(\frac{1}{n^2} + \omega_4\left(f, \frac{1}{\sqrt{n}}\right)\right). \tag{6.10}$$

Using (6.7) and (6.10), we have

$$\| B_n^{(2)}(f, x) - f(x)\|_{C(S)} \leqslant A_1\left(\frac{1}{n^2} + \omega_4\left(f, \frac{1}{\sqrt{n}}\right) + \omega_2\left(f, \frac{1}{n}\right)\right), \tag{6.11}$$

which implies (6.3).

Proof of Lemma 6.2. We shall show $p_{nk}(x) \geqslant 6p_{2n2k}(x)$ only, and other cases will follow similarly. After simple modifications one sees that we need only to show

$$A_{nk} = \frac{\binom{n}{k}}{\binom{2n}{2k}} \geqslant 6p_{nk}(x) \qquad \text{for} \quad \left| x - \frac{k}{n} \right| \geqslant A \frac{1}{\sqrt{n}}. \tag{6.12}$$

Since $p_{n,k}(x)$ has its only maximum at $x = k/n$, it is enough to show $A_{nk} \geqslant 6p_{nk}[(k/n) \pm A(1/\sqrt{n})]$. Using Stirling's formula, $A_{nk} \geqslant C\sqrt{n/(k[n-k])}$. We recall Laplace's formula of the theory of probability (see Ref. 6, p. 14, for example)

$$p_{nk}(x) \sim [2\pi x(1-x)\,n]^{-1/2} \exp\left[-\frac{n}{x(1-x)}\left(\frac{k}{n} - x\right)^2\right] \tag{6.13}$$

for $|x - (k/n)| \geqslant n^{-\alpha}\alpha > 1/3$ in particular for $x = (k/n) \pm A(1/\sqrt{n})$, from which (6.12) follows.

Remark. It was pointed out to us by G. G. Lorentz that P. Butzer considered differences of Bernstein polynomials and their rate of convergence to $f(x)$ [see *Can. Math. J.* **5** (1953), 559–567]. However, though the result there treats a more general case, for the operator given here the estimate is $\| B_n^{(2)}(f) - f \|_{C(0,1)} \leqslant Mn^{-1}\omega[f'', (1/\sqrt{n})]$ when $f'' \in C[0, 1]$, which for $\delta < x < 1 - \delta$ is not as strong an estimate as (6.3) given in Theorem 6.1 here.

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